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TORSIONAL VIBRATIONS OF A CIRCULAR DISK ON AN INFINITE TRANSVERSELY ISOTROPIC MEDIUM

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Abstract—The forced torsional vibratory motion of a rigid body with a circular base on the surface of a transversely isotropic material is investigated by using the method of Hankel transform. The infinite integral involved is evaluated through a contour integration to be discontinuous in nature.

The dynamic contact shear stress, the total applied torque, and the real-valued rotational displacement functions are calculated and expressed in terms of the frequency factor and the ratio of the anisotropic material constants. The resonant amplitudes and frequencies of vibration are shown to depend on the anisotropic material constants and the mass ratio.

INTRODUCTION

The forced vertical vibration of a rigid body with a circular base on an infinite, transversely isotropic material was investigated recently with the Hankel transform techniques and contour integration (Tsai, 1988). The resonant amplitude was shown to depend strongly on the anisotropic material constants, the mass ratio, and the vibration frequency. Many fiber-reinforced composite materials and platelet systems are described as transversely isotropic materials and have five elastic constants (Christensen, 1979; Postma, 1955). A layered system of typical earth materials, such as limestone and sandstone, was also described as a transversely isotropic material (Postma, 1955).

The forced torsional vibration of a body with a circular base on an infinite isotropic material has been shown to depend on the forcing frequency and the medium properties (Arnold *et al.*, 1955). The forced vibration problem for a body with a circular base has its application both in the vibrations of a machine foundation (Arnold *et al.*, 1955) and in the earthquake research on the dynamic soil-structure interaction (Luco and Westman, 1971).

The widespread use of composite materials in structural applications has generated considerable interest in the behavior of anisotropic materials. The forced torsional vibration of a rigid body with a circular base on an infinite, transversely isotropic material is investigated in the present work. The method of Hankel transform is used to solve the equation of motion and satisfy the mixed boundary conditions. The infinite integral involved is evaluated through a contour integration. The results reveal the discontinuous nature of the infinite integral.

The dynamic contact shear stress and the total applied torque are shown to be dependent on the forcing frequency and the anisotropic material constants. The real-valued displacement functions are obtained and calculated numerically for different values of frequency factor and sample material constants. The resonant amplitude and frequency are also investigated in terms of the mass ratio and the material anisotropy.

BASIC EQUATIONS

The stress–strain relationship in cylindrical coordinates (r, θ, z) for a transversely isotropic medium can be written in the following form (Tsai, 1988; Christensen, 1979; Postma, 1955):

$$\sigma_{rr} = c_{11}\, e_{rr} + c_{12}\, e_{\theta\theta} + c_{13}\, e_{zz}$$

$$\sigma_{\theta\theta} = c_{12}\, e_{rr} + c_{11}\, e_{\theta\theta} + c_{13}\, e_{zz}$$

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$$\sigma_{zz} = c_{13}\, e_{rr} + c_{13}\, e_{\theta\theta} + c_{33}\, e_{zz}$$
$$\sigma_{rz} = c_{44}\, e_{rz}, \quad \sigma_{\theta z} = c_{44}\, e_{\theta z}$$
$$\sigma_{r\theta} = (c_{11} - c_{12})\, e_{r\theta}/2. \tag{1}$$

The z-axis is along the axis of symmetry of the material. The circular base with radius a of a rigid body is assumed to rest on the free surface of the medium at $z = 0$. The base undergoes a torsional vibration about the z-axis with a circular frequency ω. The displacement field of the medium's dynamic response to the vibration can be described as $(0, U_\theta, 0)$. In response to the forced vibrations, the circumferential displacement can be written as

$$U_\theta = v\, e^{i\omega\tau} \tag{2}$$

where τ is the time variable. If the strain–displacement relations and the stress–strain relations in eqn (1) are used, the equations of motion have only one nonvanishing component as follows.

$$\frac{1}{2}(c_{11} - c_{12})\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rU_\theta)\right] + c_{44}\frac{\partial^2 U_\theta}{\partial z^2} = \Delta\frac{\partial^2 U_\theta}{\partial \tau^2} \tag{3}$$

where Δ is the density of the medium. If the first-order Hankel transform is applied to eqn. (3) over the variable r, the transformed equation in terms of the parameter s has the following form:

$$\frac{\partial^2 \bar{v}^1}{\partial z^2} - \delta^2 g^2 \bar{v}^1 = 0 \tag{4}$$

$$\delta^2 = (c_{11} - c_{12})/2c_{44}, \quad g^2 = s^2 - \omega^2/(c_2\delta)^2, \quad c_2^2 = c_{44}/\Delta \tag{5}$$

where $\bar{v}^1$ is the first-order Hankel transform of v.

The boundary condition and the contact stress on the free surface $z = 0$ can be written as

$$U_\theta = -r\theta_0\, e^{i\omega t} \quad r \leqslant a \tag{6}$$

$$\sigma_{z\theta} = \begin{cases} \tau(r, \omega) & r \leqslant a \\ 0 & r > a \end{cases} \tag{7}$$

where θ_0 is the amplitude of the rotation of the disk about the vertical z-axis. The unknown contact shear stress $\tau(r, \omega)$ is to be determined later. For the boundary conditions prescribed, the solution of the transformed displacement in eqn (4) can be written as

$$\bar{v}^1 = A\, e^{-\delta g z}. \tag{8}$$

The shear stress $\sigma_{\theta z}$ in eqn (1) is calculated in terms of the displacement. The stress boundary condition in eqn (7) determines the unknown quantity in eqn (8) as

$$A = -\bar{\tau}^1/c_{44}\delta g, \quad \bar{\tau}^1 = \int_0^a rJ_1(sr)\tau\, dr \tag{9}$$

where $\bar{\tau}^1$ is the first-order Hankel transform of the boundary shear stress in eqn (7).

INTEGRAL EQUATION

The rotational displacement is calculated from eqn (8) in terms of eqn (9); it can be written at $z = 0$ in the following form:

$$v = -\frac{1}{c_{44}\delta}\int_0^\infty J_1(sr)\hat{\tau}^1\,\mathrm{d}s - \frac{1}{c_{44}\delta}\int_0^\infty J_1(sr)\left(\frac{s}{g}-1\right)\hat{\tau}^1\,\mathrm{d}s. \tag{10}$$

The first-order Bessel function $J_1(x)$ results from the inverse transform of eqn (8). If the frequency approaches zero, the second term on the right-hand side of eqn (10) vanishes. In other words, the first term on the right-hand side is the associated static circumferential displacement.

To solve for the unknown contact shear stress τ in eqn (10), the transform shear stress is written in terms of a parameter function as follows:

$$\hat{\tau} = c_{44}\delta\int_0^a h(t^2)\sin(st)\,\mathrm{d}t. \tag{11}$$

If (11) is substituted into (10), the circumferential displacement for $r \leqslant a$ becomes

$$v = -\int_0^r h(t^2)\frac{t\,\mathrm{d}t}{r\sqrt{r^2-t^2}} - \frac{2}{\pi}\frac{1}{r}\int_0^a h(t^2)\int_0^r \frac{\lambda}{\sqrt{r^2-\lambda^2}} I(\lambda,t)\,\mathrm{d}\lambda\,\mathrm{d}t \tag{12}$$

$$I(\lambda, t) = \int_0^\infty \sin(s\lambda)\sin(st)(s/g-1)\,\mathrm{d}s. \tag{13}$$

The first term on the right-hand side of eqn (12) results from the integration over the parameter s (Watson, 1966). The second term on the right-hand side is obtained after this identity was used:

$$J_1(sr) = \frac{2}{\pi}\frac{1}{r}\int_0^r \eta\sin(\eta s)(r^2-\eta^2)^{-1/2}\,\mathrm{d}\eta. \tag{14}$$

The parameter function h is solved from the first term on the right-hand side of eqn (12), and the result can be written as

$$h(\eta^2) = h_0 - \frac{2}{\pi}\int_0^a h(t^2)I(\eta,t)\,\mathrm{d}t \tag{15}$$

$$h_0 = -\frac{2}{\pi}\frac{1}{\eta}\frac{\partial}{\partial\eta}\int_0^\eta \frac{r^2 v}{\sqrt{\eta^2-r^2}}\,\mathrm{d}r. \tag{16}$$

The operation on the right-hand side of eqn (16) indicates the procedure to solve for h from (12). The second term on the right-hand side of (15) is obtained after integration by parts over the variable r. The value of v in (16) can be seen from (6) to be equal to $-r\theta_0$ for $r \leqslant a$. Equation (15) is a Fredholm integral equation of the second kind with the unknown $h(\eta^2)$.

CONTACT TORQUE AND RESONANCE

The contact shear stress is calculated through the inverse transform of eqn (11). The infinite integration over the parameter s in the inversion integral is carried out (Watson, 1966) and the contact shear stress for $r \leqslant a$ has the following form

$$\tau = -c_{44}\delta \frac{\partial}{\partial r}\int_r^a \frac{h(t^2)}{\sqrt{t^2-r^2}}\,\mathrm{d}t. \tag{17}$$

The contact shear stress vanishes for $r > a$. This satisfies the prescribed condition in eqn (7). The total torque exerted between the disk and the medium is calculated from (17) as follows.

$$M = 2\pi\int_0^a \tau r^2\,\mathrm{d}r = 4\pi c_{44}\delta\int_0^a h(t^2)t\,\mathrm{d}t. \tag{18}$$

To solve for the parameter function h, the displacement function in (6) is substituted into (16). The result after integrations is

$$h_0 = 4\theta_0\eta/\pi. \tag{19}$$

The integral I in eqns (13) and (15) is evaluated by the techniques of contour integration (Tsai, 1988; Christensen, 1979; Postma, 1955) to have two different forms of expression as follows.

$$I(\eta,t) = \begin{cases} \displaystyle\int_0^\beta s(\beta^2-s^2)^{-1/2}\,e^{-ist}\sin(s\eta)\,\mathrm{d}s, & t > \eta \\ \displaystyle\int_0^\beta s(\beta^2-s^2)^{-1/2}\,e^{-is\eta}\sin(st)\,\mathrm{d}s, & t < \eta \end{cases} \tag{20}$$

where $\beta = \omega/c_2\delta$. The integrations in eqn (15) are all of finite range. In view of the above results, the parameter function h in (15) is a complex-valued function. If the transformations $s = \beta\zeta$, $\eta = an$, and $t = am$ are introduced, eqn (20) can be written after integrations in the following forms.

$$I(\eta,t) = -N(n,m)/a; \quad N = N_1 + iN_2 \tag{21}$$

$$4N_1(n,m)/\pi = \begin{cases} J_1[k(n-m)/\delta] - J_1[k(n+m)/\delta], & n > m \\ J_1[k(m-n)/\delta] - J_1[k(n+m)/\delta], & n < m \end{cases} \tag{22}$$

$$N_2(n,m) = \int_0^1 \zeta(1-\zeta)^{-1/2}\sin(km\zeta/\delta)\sin(kn\zeta/\delta)\,\mathrm{d}\zeta \tag{23}$$

where the frequency factor is $k = \omega a/c_2$. To obtain nondimensional forms of solution, the parameter function is normalized as $h(\eta^2) = 4\theta_0 a\bar{h}(n^2)\pi$. The integral equation (15) is reduced to the nondimensional form as follows:

$$\bar{h}(n^2) = n + \frac{2}{\pi}\frac{k}{\delta}\int_0^1 \bar{h}(m^2)N(n,m)\,\mathrm{d}m. \tag{24}$$

In terms of the nondimensionalized parameter function $\bar{h}(n^2)$, the contact torque in eqn (18) can be written as

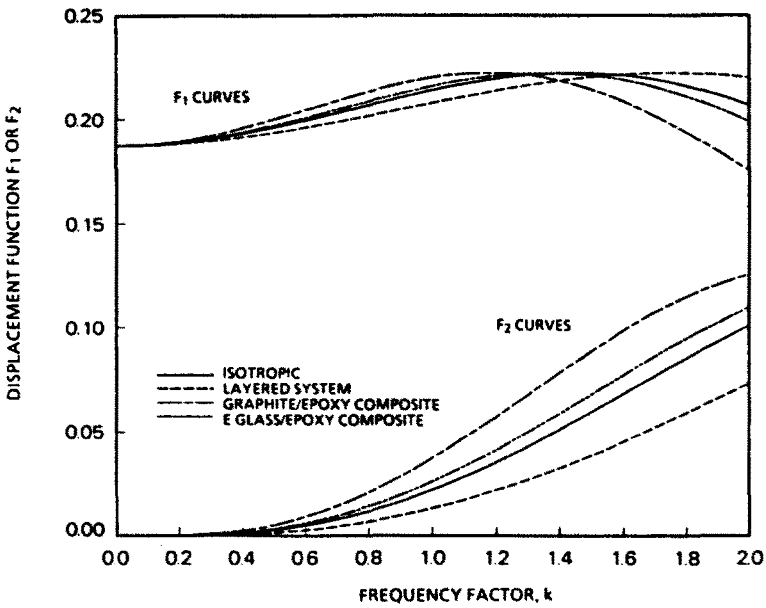

Fig. 1. The rotational displacement functions F_1 and F_2.

$$M = c_{44}\delta a^3 T\theta_0 \tag{25}$$

$$T = T_1 + iT_2 = 16\int_0^1 \bar{h}m \, \mathrm{d}m. \tag{26}$$

Equation (25) has the following alternative form:

$$c_{44}\delta a^3\theta_0/M = F_1 + iF_2 \tag{27}$$

where F_1 and F_2 are the real-valued displacement functions which result from the inverse of the integral over $\bar{h}$ in eqn (26).

Composite materials and a layered geological system are used as example materials for the study of the combined effects of the material anisotropy and the forcing frequency. Graphite/epoxy and E glass/epoxy composites have been described as transversely isotropic materials (Behrens, 1971). The material constants for graphite/epoxy composite are $c_{11} = 0.828$, $c_{33} = 8.68$, $c_{13} = 0.0285$, $c_{12} = 0.2767$, and $c_{44} = 0.4147$; for E glass/epoxy composite they are $c_{11} = 1.493$, $c_{33} = 4.727$, $c_{13} = 0.5244$, $c_{12} = 0.6567$, and $c_{44} = 0.4745$, all in the unit of 10^4 MN/m^2. The constants for the limestone/sandstone layered system in 10^4 MN/m^2 are $c_{11} = 6.25$, $c_{33} = 4.57$, $c_{13} = 1.74$, $c_{12} = 2.19$ and $c_{44} = 1.4$ (Postma, 1955).

The nondimensional parameter function $\bar{h}(\lambda^2)$ is solved from the Fredholm integral equation in (24) by using the numerical procedures devised in Baker *et al.* (1964). For an isotropic material, c_{44} is equal to the shear modulus μ, and the value of δ reduces to unity. The values of F_1 and F_2 in eqn (27) become independent of material constants (as was pointed out in Arnold *et al.* (1955) for an isotropic material). The values of F_1 and F_2 are calculated in terms of $\bar{h}$ and depend on anisotropic material constants, as can be seen in Fig. 1. The values of F_1 and F_2 for an isotropic material are in agreement with the corresponding graphical value in Arnold *et al.* (1955).

The mass moment of inertia of the vibrating rigid body about the vertical axis, I_0, is often considered in practical applications. Equation (25) can be used to incorporate the effects of the rotational inertia and yields the following nondimensional form (Tsai, 1988; Arnold *et al.*, 1955).

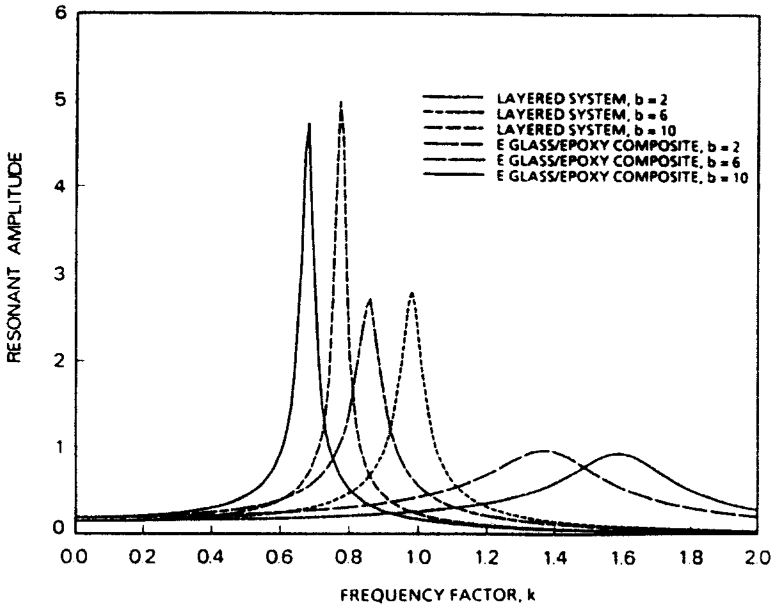

Fig. 2. The torsional resonant amplitude and frequency.

$$|c_{44}a^3\theta_0/M| = [(T_1 - bk^2/\delta)^2 + T_2^2]^{-1/2}/\delta \tag{28}$$

where the mass ratio is $b = I_0/\Delta a^5$, with Δ as the medium density. For an isotropic material, the resonant effects predicted by (28) are in agreement with the experimental results shown in Arnold *et al.* (1955).

The resonant amplitudes for the E glass/epoxy composite and the layered system of limestone and sandstone are shown in Fig. 2 for various values of mass ratio and frequency factors.

CONTACT SHEAR STRESS

The shear stress in the contact area in eqn (17) is calculated to have the following form.

$$\tau = c_{44}\delta\left\{\frac{r}{\sqrt{a^2-r^2}}\frac{h(a)}{a} - \int_r^a \frac{r}{\sqrt{t^2-r^2}}\frac{\partial}{\partial t}\left[\frac{h(t^2)}{t}\right]\mathrm{d}t\right\}. \tag{29}$$

In terms of eqns (15) and (21)–(23), the derivative of the parameter function is written as

$$\frac{\partial}{\partial\eta}h(\eta^2) = \frac{4}{\pi}\theta_0 + \frac{2}{\pi}\int_0^a h(t^2)\int_0^\beta \frac{s^2}{\sqrt{\beta^2-s^2}}L(\eta,s,t)\,\mathrm{d}s\,\mathrm{d}t \tag{30}$$

$$L = L_1(\eta, st) + iL_2(\eta, st) \tag{31}$$

$$L_1 = \begin{cases}\sin(s\eta)\sin(st), & t < \eta\\ \cos(s\eta)\cos(st), & t > \eta\end{cases} \tag{32}$$

$$L_2 = \cos(s\eta)\sin(st). \tag{33}$$

The differentiation in the second term on the right-hand side of eqn (29) is calculated in terms of eqns (15) and (30). After normalization, the contact shear stress becomes

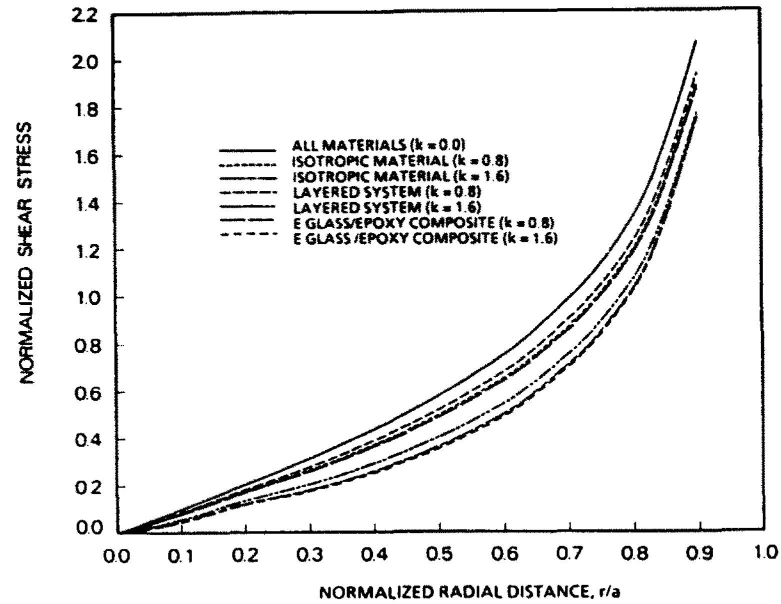

Fig. 3. The normalized contact shear stress.

$$\tau = c_{44}\delta\frac{4}{\pi}\theta_0\left\{\frac{\rho}{\sqrt{1-\rho^2}}\bar{h}(1)-\frac{2}{\pi}\frac{1}{\delta}k\int_\rho^1\frac{\rho}{\sqrt{n^2-\rho^2}}H(n)\,\mathrm{d}n\right\} \tag{34}$$

$$H(n) = \int_0^1 h(m^2)\int_0^1\frac{\zeta}{\sqrt{1-\zeta^2}}Q(n,\zeta,m)\,\mathrm{d}\zeta\,\mathrm{d}m \tag{35}$$

$$Q = Q_1(n,\zeta,m)+iQ_2(n,\zeta,m) \tag{36}$$

$$Q_1 = \begin{cases}[k\zeta n\sin(k\zeta n)+\cos(k\zeta n)]\sin(k\zeta m)/n^2 & (37)\\ [\sin(k\zeta n)-k\zeta n\cos(k\zeta n)]\cos(k\zeta m)/n^2 & (38)\end{cases}$$

$$Q_2 = [k\zeta n\cos(k\zeta n)-\sin(k\zeta n)]\sin(k\zeta m)/n^2 \tag{39}$$

where $\rho = r/a$. If the forcing frequency tends to zero, the second term on the right-hand side of (34) vanishes. In other words, the first term on the right-hand side of (34) is the associated static contact stress.

The normalized contact shear stress $\tau/(4c_{44}\delta\theta_0/\pi)$ is calculated as a function of the frequency factor and the material constants. Typical values are shown in Fig. 3.

DISCUSSION AND CONCLUSION

The torsional vibrations of a rigid body with a circular base on the surface of an infinite transversely isotropic material are investigated by using the method of Hankel transform. An infinite integral involved in the process of solution is evaluated through a contour integration and shown to be discontinuous in nature. The dynamic contact shear stress is shown to be a function of the forcing frequency and the anisotropic material constants. The normalized dynamic contact stress is seen in Fig. 3 to decrease with increasing frequency.

The total dynamic torque is calculated from the contact stress. The real-valued displacement functions, which account for the normalized ratio between the forcing rotational displacement and the total torque, are expressed in terms of the frequency factor and the material constant ratio $\delta = (c_{11}—c_{12})/2c_{44}$. The values of the functions are shown in Fig. 1 for four different sample materials. The F_2 curves for the graphite/epoxy and E glass/epoxy composites are higher than the curve for the isotropic material. However, the F_2 curve for the layered system of limestone and sandstone is seen in Fig. 1 to be lower than that for the isotropic material. The F_1 curves have features similar to those for the F_2 curves at low frequencies. At relatively higher frequencies, the relative positions of the F_1 curves are reversed (Fig. 1).

The resonant amplitudes and frequencies shown in Fig. 2 depend on the material constants, the mass ratio b, and the vibration frequency. The resonant amplitude increases while the resonant frequency decreases if the value of the mass ratio increases. The resonance of the layered system occurs at a relatively higher frequency than that for the E glass/epoxy composite for the same mass ratio.

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